Indian Statistical Institute

Mid-Semestral Examination 2013-2014

M.Math First Year

Complex Analysis

Time : 3 Hours   Date : 03.03.2014   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. (iii) $\mathbb{H}$ = the upper half plane.

Q1. (15 marks) Let f be a non-vanishing continuous function on an open set $U \subseteq \mathbb{C}$ and assume that f^2 is holomorphic in U. Prove that f is holomorphic in U. Compute the derivative of f.

Q2. (15 marks) Let f be an entire function. True or False (with justification)?

(i) if $f(\mathbb{C}) \subseteq \mathbb{C} \setminus \mathbb{D}$, then f is constant.

(ii) If $f(\mathbb{C}) \subseteq \mathbb{H}$, then f is constant.

Q3. (15 marks) Prove that there exists a constant $c > 0$ such that

$$\sup\{|\frac{1}{z} - p(z)| : |z| = 1\} > c,$$

for all $p \in \mathbb{C}[z]$.

Q4. (15 marks) Let γ be a smooth curve in $\mathbb{H}$ joining $-1+i$ to $1+2i$. Find

$$\int_\gamma \frac{dz}{1+z}.$$

Q5. (15 marks) Let f be an entire function and $f(x,y) = u(x) + iv(y)$ for all $x, y \in \mathbb{R}$. Prove that f is a polynomial. Compute the degree of f.

Q6. (9+ 5+ 6 = 20 marks) Let f be a non-constant holomorphic function on an open set $U \supseteq \overline{\mathbb{D}}$ and $|f(z)| = 1$ for all $z \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Prove that

(i) $f(z) = 0$ for some $z \in \mathbb{D}$.

(ii) $|\varphi_\alpha(z)| = 1$ for all $z \in \mathbb{T}$. Here $\alpha \in \mathbb{D}$ and $\varphi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha}z}$ and $z \in \mathbb{D}$.

(ii) For each $\alpha \in \mathbb{D}$, $f(z) = \alpha$ for some $z \in \mathbb{D}$.

Q7. (15 marks) Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be a power series which converges on $\mathbb{D}$ and $f(\mathbb{D}) \subseteq \overline{\mathbb{D}}$. Prove that $|a_n| \leq 1$ for all $n \geq 0$.